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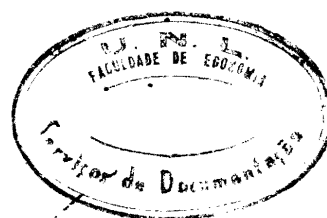
IMPROVING LAGRANGEAN DECOMPOSITION:

SOME THEORETICAL RESULTS

Paulo Bárcia

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Introdução

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IMPROVING LAGRANGEAN DECOMPOSITION :
SOME THEORETICAL RESULTS

by

PAULO BARCIA (*)

(*)Universidade Nova de Lisboa, Faculdade de Economia, Tv. Estevão
Pinto , Campolide , P-1000 LISBOA , Portugal.

ABSTRACT

Recently two new ways of obtaining improved Lagrangean bounds have been suggested : Variable splitting / Lagrangean decomposition and bound improving sequences.

The aim of this work is to obtain a Lagrangean approach combining the two ideas mentioned above.

We provide some theoretical results about the sharpness of the bounds obtained by the combined approach. We show that they dominate the bounds obtained by any of the two individual techniques.

1. INTRODUCTION

Recently two new ways of obtaining improved Lagrangean bounds have been suggested : Lagrangean decomposition ,Jörnsten Näsberg & Smeds (1985) and Guignard & Kim (1987)' , and bound improving sequences, Barcia (1987) .

In this work we follow the path layed down in Barcia & Jörnsten (1986) to obtain a Lagrangean approach combining the two ideas mentioned above.

In section 2 we will briefly sketch the bound improving sequence technique which consists in building a sequence of Lagrangean duals that progressively reduces the duality gap and , if some conditions are met, converges in a finite number of steps to the optimal value of the original problem .

Section 3 will be devoted to the discussion of the main results concerning Lagrangean decomposition / Variable splitting which consists in duplicating the problem variables thus enabling the use of more than one structure within a Lagrangean approach. In section 4 we shall , following the ideas layed down in Barcia & Jörnsten (1986) ,study the combination of the two approaches mentioned above and provide some theoretical results about the bounds thus obtained for the combined approach.

Finally in section 5 we shall draw some conclusions..

2. BOUND IMPROVING SEQUENCES

In this section we briefly sketch the basics of the bound improving sequence technique. We shall state only the main results and the reader is referred to Barcia(1985) , Barcia(1987) and Barcia & Holm(1988) for formal proofs and more details on the subject.

Now consider the following integer linear programming problem:

$$\begin{aligned} (P) \quad & z = \min cx \\ & Ax \leq b \\ & x \in \mathbb{X} \end{aligned}$$

where c is an integer n -vector, b an integer m -vector and A a matrix of integers of appropriate dimension. For simplicity we shall assume that $\mathbb{X} \subset \mathbb{Z}^n$ is bounded and that problem (P) has always a finite solution.

Suppose now that a lower bound for z , say $\ell_k \leq z$, is known. Defining the set $\mathbb{X}_k = \{ x \in \mathbb{X} : cx \geq \ell_k \}$ problem (P) can now be restated in the following equivalent way:

$$\begin{aligned} (Pk) \quad & z = \min cx \\ & Ax \leq b \\ & x \in \mathbb{X}_k \end{aligned}$$

Now take $u \in \mathbb{R}_+^m$ as a vector of non-negative multipliers and consider a Lagrangean dual of (Pk) defined as follows:

$$(LDk) \quad \ell_{k+1} = \max_{u \in \mathbb{R}_+^m} \min_{x \in \mathbb{X}_k} cx + u(Ax - b) \quad k=0,1,\dots$$

Note first that the usual Lagrangean bound can be obtained from (LDk) as $\ell = \ell_1$ by taking $\ell_0 = -\infty$. For simplicity we shall assume that ℓ_0 was selected this way and we mention by ℓ the usual Lagrangean bound whenever we need to refer it.

Note also that, if $\mathbb{X} = \{0,1\}^n$, the inner minimization problem on the right hand side of the equality is a single 0-1 knapsack problem which, although in \mathcal{NP} , is one of the easiest non-polynomial problems, see Dudzinsky & Walukiewicz (1987). Since the outer maximization problem can be solved using a subgradient algorithm, see Demianov & Vasilev (1985), (LDk) can be computed with a moderate effort.

Note also that if \mathbb{X} is somewhat more complicated, say $\{0,1\}^n$ with semi-assignment constraints, one would get multiple-choice knapsack problems which are as easy as single 0-1 knapsack problems, see Dudzinsky & Walukiewicz (1987). So, some structure may be kept in the \mathbb{X} set without deteriorating the computational efficiency of the method.

One can easily prove the following :

Theorem 2.1 : $\ell_{k+1} \geq \ell_k$

Proof: See Barcia (1985). ■

Some necessary conditions for equality to hold in theorem 2.1

are easily derived . For instance , one has :

Theorem 2.2 : If $\ell_{k+1} = \ell_k$ then $u=0$ is an optimal vector of multipliers in (LDk).

•Proof: See Barcia (1985). ■

Of course the interesting question is to be able to characterize the situations such that the above inequality is strict . The answer for this is addressed in the following:

Theorem 2.3 : Consider the level sets of the objective function, $L(y) = \{ x \in X : cx = y \}$. Then $\ell_{k+1} > \ell_k$ if and only if $\text{Conv } L(\ell_k) \cap \{ x \in \mathbb{R}^n : Ax \leq b \} = \emptyset$.

•Proof: See Barcia (1987). ■

Note now that if the objective function cx is such that its level sets are singletons the first value on the sequence $\{\ell_k\}$ for which theorem 2.2 does not hold will be z . In this case (LDk) provides a device to bridge the duality gap in a finite number of steps.

For a general objective function one must either supplement (LDk) with an enumeration scheme to deal with the cases for which $\ell_{k+1} = \ell_k$ or just use the above method as a device for improving Lagrangean bounds . Note that the more "complicated" the original X set is the more "likely" the level sets of cx are sigletons , so more complicated X sets will , eventually , provide better bounds

without enumeration.

When equality occurs, one must search the hyperplane $cx = \ell_k$ for a (P)-feasible point. If such a point is found it's optimal. If no such a point can be found we can take $\ell_{k+1} = \ell_k + 1$ and keep using (LDk) to generate a non-decreasing sequence of bounds converging to z .

This procedure is interesting only when enumeration is not performed very often because of its time consuming nature.

Barcia & Holm (1988) report a revised version of the basic algorithm we just sketched enabling some savings in the number of times that the enumeration scheme must be used.

In this note we shall look at (LDk) just as a device for improving available bounds, so no enumeration step will be needed.

Let us state now a final result in this section which, although not computationally interesting, will be very useful when trying to understand, from a primal standpoint, how (LDk) operates:

Theorem 2.4: Computing (LDk) is tantamount to convexifying the nonrelaxed constraints in problem (Pk), i.e., we have :

$$\begin{aligned} \ell_{k+1} &= \min cx \\ Ax &\leq b \\ x &\in \text{Conv } \mathcal{X}_k \end{aligned}$$

Proof: See Barcia (1987). ■

3. LAGRANGEAN DECOMPOSITION

In this section we recall and comment the main results on the Variable splitting / Lagrangean decomposition approach. Our aim is to lay the ground for section 4 where we will use these results to show how a combined approach with bound improving sequences can provide bounds dominating those obtained by any of the two individual techniques.

Again, only the main ideas will be sketched. All proofs will be omitted. The reader is referred to Jörnsten Näsberg & Smeds (1985) and Guignard & Kim (1987) for full proofs and a more detailed treatment of the subject. Some previous applications of the technique can be found in Ribeiro & Minoux (1984), Jörnsten & Näsberg (1986) and Minoux (1987), among others.

Let us consider a structured pure integer programming problem of the form

$$\begin{array}{ll}
 & z = \min \quad cx \\
 \text{(SP)} & Ax \leq b \\
 & Bx \leq d \\
 & x \in \mathbb{X}
 \end{array}$$

in which the constraint matrix A is such that a problem containing only the constraints $\{ Ax \leq b, x \in \mathbb{X} \}$ is easier to solve than a general integer programming problem, i.e., there exists some special purpose method for the problem $\{\min cx : Ax \leq b, x \in \mathbb{X}\}$. We will also assume that the same is true for the problem in the

second structure $\{ \min cx : Bx \leq d, x \in X \}$. However problem (SP) in which both constraint sets are present is assumed to be much more difficult to solve.

Problems involving such usable substructures are often solved using a Lagrangean relaxation approach in which one of the constraint sets is relaxed thus originating subproblems in the other substructure as follows :

$$\begin{aligned} \ell_A = \max_{u \geq 0} \min_{\substack{Bx \leq d \\ x \in X}} cx + u (Ax - b) \end{aligned}$$

Depending on the problem and on the constraint sets one of the bounds ℓ_A or ℓ_B , where ℓ_B stands for the Lagrangean bound obtained by relaxing only the constraints $Bx \leq d$, may be stronger than the other.

Recently a method that makes use of more than one structure has been suggested and named "variable splitting" by Jörnsten Näsberg & Smeds (1985) or "Lagrangean decomposition" by Guignard & Kim (1987).

The idea behind the technique is to use a different 'copy' of the original variables for each substructure and thus reformulate (SP) into a problem having twice as many variables as in problem (SPR) below:

$$\begin{aligned}
 z &= \min(\alpha cx + \beta cy) \\
 Ax &\leq b \\
 By &\leq d \\
 x &= y \\
 x &\in X, y \in Y
 \end{aligned}
 \tag{SPR}$$

which is a valid reformulation for any $Y \supset X$ and any $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta = 1$.

If the constraints $x = y$ are relaxed the subproblem obtained will be separable, i.e., in order to solve the Lagrangean dual one has to solve two subproblems with known usable structures.

Consider then free Lagrangean multipliers $v \in \mathbb{R}^n$ for the equality constraints $x = y$. The Lagrangean dual of (SPR) will be:

$$\begin{aligned}
 \text{(LDR)} \quad d &= \max_{v \in \mathbb{R}^n} \min [(\alpha c - v)x + (\beta c + v)y] \\
 &\quad Ax \leq b \quad By \leq d \\
 &\quad x \in X \quad y \in Y
 \end{aligned}$$

Note that we have now a x -subproblem and a y -subproblem taking full advantage of the two substructures exhibited by the problem. Note also that, as we must only have $Y \supset X$, one may "forget" the integrality constraints in one of the subproblems.

So the new bound d is easy to compute. Of course the key issue is to know how does it relates to the usual Lagrangean bounds ℓ_A and ℓ_B . The following result can be proved:

Theorem 3.1: $d \geq \max \{ \ell_A, \ell_B \} \geq \ell$.

Proof: See Jörnsten Näsberg & Smeds (1985). ■

So we know that the Lagrangean decomposition bound dominates the usual Lagrangean bounds.

Of course the interesting question is to be able to tell in which situations the above inequality degenerates into a equality. The answer for this is based on a property which is an analogue of Geoffrion's integrality property for the conventional Lagrangean technique:

Definition 3.2: The set $\{ x : Ax \leq b \}$ is said to be \mathcal{X} -convex if $\text{Conv} \{ x : Ax \leq b, x \in \mathcal{X} \} = \{ x : Ax \leq b, x \in \text{Conv } \mathcal{X} \}$.

Using the above \mathcal{X} -convexity property we can now state a sufficient condition for equality to hold in theorem 3.1. In fact, the following result holds:

Theorem 3.3: If the set $\{ x : Ax \leq b \}$ is \mathcal{X} -convex and if $\{ y : By \leq d, y \in \mathcal{Y} \}$ is compact then $d = \ell_A$.

Proof: See Guignard & Kim (1987). ■

Note first that a similar result holds for the case of the equality $d = \ell_B$. It can be derived from theorem 3.3 simply by interchanging the roles of the two constraint sets.

Note also that the compacticity hypothesis assumed above is

very "mild" since it will be always verified if we take $Y = X$ as X is assumed to be a bounded subset of Z^n . This means that, for "practical" purposes, the X -convexity property implies the equality $d = \ell_A$.

We shall now terminate this section by stating a result, similar in nature to theorem 2.4, giving a nice primal interpretation of (LDR) that will be useful later on.

Theorem 3.4: Computing (LDR) is tantamount to minimizing cx over the intersection of the convex hulls of the two substructures in (SP), i.e., we have:

$$d = \{ \min cx : x \in \text{Conv}\{x \in X : Ax \leq b\} \cap \text{Conv}\{x \in Y : Bx \leq d\} \}$$

Proof: See Guignard & Kim (1987). ■

4. THE COMBINED APPROACH

In this section we shall combine the ideas of Lagrangean decomposition with those of bound improving sequences in order to obtain a bound that dominates the one obtained by any of the two individual approaches.

As in section 3 we shall consider the specially structured pure integer programming problem (SP).

Suppose now that a lower bound for z , the optimal value of (SP), is available and call it d_k . As in section 2 let us now consider the set $X_k = \{ x \in X : cx \geq d_k \}$.

We can now state the following problem

$$\begin{aligned}
 (SPR_k) \quad & z = \min(\alpha cx + \beta cy) \\
 & Ax \leq b \\
 & By \leq d \\
 & x = y \\
 & x \in X_k, y \in Y
 \end{aligned}$$

which we know to be a valid reformulation of (SP) for any $Y \supset X_k$ and any $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta = 1$.

In order to simplify some proofs we shall assume that the starting lower bound used was the Lagrangean decomposition bound, i.e., $d_0 = d$. We shall further assume that we took $\alpha=1$ and $\beta=0$ and that Y was chosen in such a way that the set $\{ y \in Y : By \leq d \}$ is a non-empty compact of \mathbb{R}^n .

Now take a Lagrangean dual of (SPR_k) by relaxing the equality

constraints $x = y$. One gets:

$$(LDRk) \quad d_{k+1} = \max_{v \in \mathbb{R}^n} \min_{\substack{Ax \leq b \\ x \in X_k}} [(c-v)x + vy] \quad \text{By} \leq d \quad y \in Y$$

Note that (LDRk) defines , for $k=0,1,\dots$, a sequence of bounds in the spirit of section 2 .Note also that d_{k+1} need not be more difficult to compute than the usual Lagrangean decomposition bound d , depending on the structure of the constraints $Ax \leq b$ and on the original X set : if these constraints are , for instance , semi-assignment constraints (GUB constraints with RHS 1) and $X=\{0,1\}^n$ then the x-subproblem is a 0-1 multiple-choice knapsack problem which can be solved with moderate effort.

An obvious result is the following:

Theorem 4.1: $d_{k+1} \geq \ell_{k+1} \geq \ell$.

Proof: Directly from theorem 3.1.

The question we address now is the following : how does d_{k+1} relates to the usual Lagrangean decomposition bound ? . The following result is easily proved:

Theorem 4.2: $d_{k+1} \geq d_k \geq d$

Proof: The first inequality is obtained directly from (LDRk) considering the multiplier $v=0$. The second inequality is the

special case for $k=0$. ■

So now we know that (LDR $_k$) defines a bound improving sequence dominating the Lagrangean decomposition bound. The interesting issue is, of course, to be able to state a condition for strict improvement. Before addressing this question we must state and prove a very simple preliminary lemma.

Lemma 4.3: Consider S and T as two subsets of U and let $S \otimes T$ denote their Cartesian product. Then we have:

$(u,u) \in \text{Conv } S \otimes T$ if and only if $u \in \text{Conv } S \cap \text{Conv } T$.

Proof: Suppose first that $u \in \text{Conv } S \cap \text{Conv } T$. As $u \in \text{Conv } S$ we know that there must exist numbers $\alpha_i \geq 0$ and points $s_i \in S$ such that $u = \sum \alpha_i s_i$ and $\sum \alpha_i = 1$. Of course a similar property holds for points $t_j \in T$ and numbers $\tau_j \geq 0$ because u is also in $\text{Conv } T$.

Consider now all pairs $(s_i, t_j) \in S \otimes T$ and define numbers $v_{ij} \geq 0$ as $v_{ij} = \alpha_i \tau_j$. Note that $\sum v_{ij} = \sum \alpha_i (\sum \tau_j) = 1$.

Note finally that we have the following:

$$\sum v_{ij} (s_i, t_j) = (\sum \alpha_i s_i (\sum \tau_j), \sum \tau_j t_j (\sum \alpha_i)) = (u, u)$$

This means that $(u,u) \in \text{Conv } S \otimes T$ and the first part of the result follows.

Now take $(u,u) \in \text{Conv } S \otimes T$. We must then have points (s_k, t_k) in $S \otimes T$ and numbers $\mu_k \geq 0$ such that $\sum \mu_k = 1$ and $u = \sum \mu_k s_k = \sum \mu_k t_k$ which shows that $u \in \text{Conv } S \cap \text{Conv } T$ and the proof is terminated. ■

We are now equipped to prove a result that characterizes the cases for which we have $d_{k+1} > d_k$.

Theorem 4.4 : We shall have the strict inequality $d_{k+1} > d_k$ if and only if the following condition holds:

$$\text{Conv}\{x \in \mathcal{X} : cx = d_k, Ax \leq b\} \cap \text{Conv}\{y \in \mathcal{Y} : By \leq d\} = \emptyset$$

Proof: Note first that problem (SPR_k) can be stated in the following equivalent way :

$$\{ \min cx : x = y, cx \geq d_k \text{ and } (x,y) \in \mathcal{X} \}$$

where $\mathcal{X} \subseteq \mathcal{X} \otimes \mathcal{Y}$ denotes the set $\mathcal{X} = \{x \in \mathcal{X}, y \in \mathcal{Y} : Ax \leq b, By \leq d\}$.

Let's denote by \mathcal{X}_k the set $\mathcal{X}_k = \{(x,y) \in \mathcal{X} : cx \geq d_k\}$.

Now, if we relax the equality constraints $x = y$ we will obtain d_{k+1} as follows:

$$(i) \quad d_{k+1} = \max_{v \in \mathbb{R}^n} \min_{(x,y) \in \mathcal{X}_k} cx + v(y-x)$$

Note that (i) defines a bound improving sequence, as in section 2, so theorem 2.3 applies and then we know that we will have $d_{k+1} > d_k$ if and only if the level set $\mathbb{L}(d_k)$ is such that $\text{Conv } \mathbb{L}(d_k) \cap \{(x,y) \in \mathbb{R}^{2n} : x = y\} = \emptyset$. So in the convex hull of the level set $\mathbb{L}(d_k)$

$$\mathbb{L}(d_k) = \{(x,y) \in \mathcal{X} : cx = d_k\} = \{x \in \mathcal{X} : cx = d_k, Ax \leq b\} \otimes \{y \in \mathcal{Y} : By \leq d\}$$

there can not exist any point with equal coordinates $x = y$.

Then lemma 4.3 implies that we must have

$$\text{Conv}\{x \in \mathcal{X} : cx = d_k, Ax \leq b\} \cap \text{Conv}\{y \in \mathcal{Y} : By \leq d\} = \emptyset$$

and the proof is terminated. ■

Compare now the contents of theorems 3.4 and 4.4. We have the

following , because of theorem 3.4 :

$d = \{ \min cx : x \in \text{Conv}\{ x \in \mathbb{X} : Ax \leq b \} \cap \text{Conv}\{ x \in \mathbb{Y} : Bx \leq d \} \}$
and call x^* the optimal solution of the above problem . If $x^* \in \mathbb{X}$
then it's the optimal solution for the original problem and $d = z$.

Suppose now that $x^* \notin \mathbb{X}$ and let's examine the possibility of not being able to improve the usual Lagrangean decomposition bound d by using the combined approach. For simplicity we shall take $\mathbb{Y} = \mathbb{X}$.

If this is so theorem 4.4 implies the following :

$$\text{Conv}\{ x \in \mathbb{X} : cx = d , Ax \leq b \} \cap \text{Conv}\{ x \in \mathbb{X} : Bx \leq d \} \neq \emptyset$$

But this is only possible if there exist points $x_i \in \mathbb{X}$ such that $Ax_i \leq b$, $cx_i = cx^* = d$ and $x^* \in \text{Conv}\{ x_i \}$. This is a very "unlikely" situation because it would imply that $cx = d$ supports a non-zero dimensional (because $x^* \notin \mathbb{X}$) face of the politope $\text{Conv}\{ x \in \mathbb{X} : Ax \leq b \} \cap \text{Conv}\{ x \in \mathbb{X} : Bx \leq d \}$.

Excluding this case as "degenerate" we have the following result:

Corolary 4.5: Suppose $d < z$ and the following nondegeneracy assumption holds : for any $\gamma < z$ $cx = \gamma$ does not support any non-zero dimensional face of the politope

$$\text{Conv}\{ x \in \mathbb{X} : Ax \leq b \} \cap \text{Conv}\{ x \in \mathbb{X} : Bx \leq d \}.$$

Then we shall always have $d_1 > d$.

which tells us that , in almost every practical situation , the new combined approach will produce a strict improvement on the Lagrangean decomposition bound.

We shall now terminate this section by giving a primal

interpretation of (LDRk).

Theorem 4.6: Computing d_{k+1} is tantamount to solve any of the two following modified primals:

$$d_{k+1} = \min_{x=y, (x,y) \in \text{Conv } \mathcal{X}_k} cx = \min_{x \in \text{Conv } A_k \cap \text{Conv } B} cx$$

where $\mathcal{X}_k = \{ x \in \mathcal{X}, y \in \mathcal{Y} : cx \geq d_k, Ax \leq b, By \leq d \}$,
 $A_k = \{ x \in \mathcal{X} : cx \geq d_k, Ax \leq b \}$ and $B = \{ x \in \mathcal{Y} : Bx \leq d \}$.

Proof: The equality $d_{k+1} = \{ \min cx : x = y, (x,y) \in \text{Conv } \mathcal{X}_k \}$ comes from using theorem 2.4 in formulation (SPRk) while the equality $d_{k+1} = \{ \min cx : x \in \text{Conv } A_k \cap \text{Conv } B \}$ is obtained by using theorem 3.4 in the same formulation.

Note finally that the equality of the two primal formulations above can also be obtained directly using lemma 4.3. ■

6. CONCLUSIONS

In this paper we have presented how two "new" methods for pure integer programming can be combined. It has been shown that the combined method, consisting of Lagrangean decomposition and bound improving sequences ideas, has the ability to generate better bounds. For specially structured problems such as problems having semi-assignment constraints this combined method has the potential of being very efficient.

7. REFERENCES

- Barcia,P. (1985) "The bound improving sequence algorithm" ,
Operations Research Letters 4 , 27-30.
- Barcia,P. (1987) "Constructive dual methods for discrete programming",Discrete Applied Mathematics 18 , 107-117.
- Barcia,P.;Holm,S. (1988) "A revised bound improvement sequence algorithm" , European Journal of Operational Research 36 (2), 202-206.
- Barcia,P.;Jörnsten,K. (1986) " Variable splitting and constructive duality :a method for structured integer programming" , Universidade Nova de Lisboa , Faculdade de Economia , Working Paper 57 ,presented at the EURO VIII meeting , Lisbon.
- Demianov,V.;Vasilev,L. (1985) "Nonsmooth optimization" ,Optimization Software inc.,New York.
- Dudzinsky,K.;Walukiewicz,S. (1987) "Exact methods for the knapsack problem and its generalizations" , European Journal of Operational Research 28 , 3-21.
- Guignard,M.;Kim,S.(1987) "Lagrangian decomposition : a model yielding stronger Lagrangian bounds" , Mathematical Programming 39 , 215-228.
- Jörnsten,K.;Näsberg,M. (1986) " A new Lagrangian approach to the generalized assignment problem ", European Journal of Operational Research 27 , 313-323.

Jörnsten, K.; Näsberg, M.; Smeds, P. (1985) "Variable splitting : a new Lagrangean relaxation approach to some mathematical programming models" , Working paper LITH-MAT-R-85-04 , Linköping University , Sweden.

Minoux, M. (1987) "The general problem of determining an optimal intercection of two combinatorial famalies and its solution by Lagragean relaxation", in "Mathematical programming", Wiley , New York.

Ribeiro, C.; Minoux, M. (1984) "Solving hard constrained shortest path problems", presented at the ORSA/TIMS meeting , Dallas .

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